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LETTER TO THE EDITOR

Crossover between invasion percolation and the Eden model in one dimension

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Abstract. The crossover between invasion percolation (IP) and the Eden model is studied in one dimension. Although the mean run length diverges in IP, it converges in the Eden model and for a broad class of perturbations to IP. In addition, IP has anomalous fluctuation effects which are absent in the Eden model. By studying a specific family of models which interpolate between the IP ($\alpha = 0$) and Eden ($\alpha = \frac{1}{2}$) limits, we show that the behaviour of the variance crosses over to that found in the Eden model for any $\alpha > 0$.

Invasion percolation (IP) has been advanced as a model of immiscible fluid displacement in porous media (Lenormand and Bories 1980, Chandler *et al* 1982, Wilkinson and Willemsen 1983). In this model each site is assigned a random number, and at each time step the boundary site with the least random number is accepted into the cluster‖. IP is an inherently kinetic model which generates clusters that apparently have the same fractal dimension as the infinite cluster at threshold in ordinary percolation. The Eden model, on the other hand, has clusters of finite density even though it too is a kinetic growth model (Eden 1961, Richardson 1973). The growth rule in this model is that at each time step one of the boundary sites is randomly chosen and added to the cluster. Recently, Martin *et al* introduced a family of models which interpolate between IP and the Eden model and studied the crossover in two dimensions by a Monte Carlo procedure. In their model a 'temperature-like' parameter Δ describes the fluctuations of the random medium, and the limits $\Delta = 0$ and $\Delta = \infty$ correspond to IP and the Eden model respectively. Their Monte Carlo results strongly suggest that for any $\Delta > 0$ the clusters are fractal up to a cut-off length, after which they are compact. In addition, Lenormand has argued that IP scaling behaviour is observable only for very small capillary numbers.

In this letter we present an analytical study of the crossover between IP and the Eden model in one dimension (1D)¶. Although both IP and the Eden model have compact clusters in 1D, there are interesting fluctuation effects in IP that are absent in the Eden model. To be specific, the variance of the distribution and the asymptotic

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‖ A 'trapping' rule must be added to this prescription if the displaced fluid is incompressible (Chandler *et al* 1982, Wilkinson and Willemsen 1983).

¶ Analytical results on IP have also been given in two other works (Nickel and Wilkinson 1983, Chayes *et al* 1985).

value of the mean run length have qualitatively different behaviour in the two models. We shall show that for any of a large class of perturbations to the IP growth rule, the mean run length (to be defined below) tends to a finite constant, whereas this quantity diverges in IP. In addition, a family of models with a parameter α that interpolates between the IP ($\alpha = 0$) and Eden ($\alpha = \frac{1}{2}$) limits is introduced. This family will be called the α model. We demonstrate that for any $\alpha > 0$ the behaviour of the variance crosses over to that found in the Eden model. Thus, our exact results support the belief—based on Monte Carlo simulations (Martin *et al* 1984) and phenomenological arguments (Lenormand 1985)—that IP scaling behaviour is disrupted by small perturbations. Extensions of our work to higher dimensions will appear elsewhere (Nadal 1986).

In the class of models we are considering there is a fixed random number r_i associated with each site i. In IP, for example, the random numbers describe the local resistance to fluid flow. These r_i are implicitly ordered: lower values are favoured for continued growth. In general the process will depend on how the r_i are distributed. The precise form of the distribution of random values is irrelevant in IP, however, since the smallest value is always chosen (only the overall order matters, not the relative magnitudes). In the Eden model the distribution of random numbers is completely irrelevant, since the probability of selecting a site is independent of the value of its random number. Throughout this letter we assume the r_i are uniformly distributed in the interval $[0, 1]$.

We assume the growth of a cluster at a given time depends only on the values of the r_i at the boundary. Therefore the kinetics of the problem are contained in a choice function $p(r_i; B-\{r_i\})$ which gives the probability that a boundary site with random number r_i is chosen when the set of boundary values is B. The choice function will also be required to have the natural property that the probability of choosing r_i from B increases if r_i is decreased or if one or more of the other boundary values are increased.

Once the distribution of site values and the choice function p are given, the growth process is completely specified for any initial set of occupied sites. IP has the choice function $p(r_i; B-\{r_i\}) = \Pi_{r_j \in B-\{r_j\}} \Theta(r_j - r_i)$, where Θ is the Heaviside distribution. The Eden model is obtained for $p(r_i; B-\{r_i\}) = |B|^{-1}$, where $|B|$ is the number of sites in B.

In 1D these growth models become particularly simple. There are only two boundary sites so the choice function can be written as $p(r_1, r_2)$, the probability that r_1 is chosen when the site values are r_1 and r_2. Our monotonicity assumption means that p must be a decreasing function of r_1 and an increasing function of r_2. It will be useful to define $\Phi(r) \equiv \int dr' p(r', r)$, the probability that the value r is not chosen. Note that $\Phi(r)$ is an increasing function and that $\Phi(r) = r$ for IP and $\Phi(r) = \frac{1}{2}$ for the Eden model.

The growth of the cluster may be broken down into a series of runs in which sites are repeatedly selected on one side of the cluster. During such a run, the values chosen are all compared with the same value. The value of the kth distinct random number which is rejected will be called the kth stopping value and will be denoted by r_k. The kth run ends when a value on the growing side is compared with r_k and is not chosen. The rejected value then becomes r_{k+1} and the cluster then grows on the opposite side from its starting point at the random number r_k. Note that in IP the sequence $\{r_k\}$ is strictly increasing. The probability distribution for the kth stopping value will be denoted $\Pi_k(r)$, and the length of the kth run will be denoted by n_k. We will show that the mean run length $\langle n_k \rangle$ diverges as $k \to \infty$ in IP, whereas if the IP growth rule is changed slightly $\langle n_\infty \rangle \equiv \lim_{k\to\infty} \langle n_k \rangle$ is finite.

To study $\langle n_k \rangle$ we first develop a recursion relation for $\Pi_k(r)$. Note that if r' is the kth stopping value, then the kth run is a succession of independent events in which

r' is not chosen, followed by one event in which the $(k+1)$th stopping value r is chosen over r'. Summing over the possible run lengths we obtain

$$\Pi_{k+1}(r)=\int_0^1 \mathrm{d}r'\Pi_k(r')p(r',r)\sum_{n=0}^{\infty}\Phi^n(r')=\int_0^1 \mathrm{d}r'\frac{\Pi_k(r')p(r',r)}{1-\Phi(r')}. \qquad (1)$$

The mean run length of the kth run is then

$$\langle n_k\rangle=\sum_{n=0}^{\infty}\int_0^1 \Pi_k(r)(n+1)\Phi^n(r)[1-\Phi(r)]\,\mathrm{d}r=\int_0^1\frac{\Pi_k(r)}{1-\Phi(r)}\,\mathrm{d}r. \qquad (2)$$

The Π_k are easily evaluated in IP and the Eden model. Suppose that only the origin is occupied at time $t=0$. For IP the initial distribution is then $\Pi_1(r)=2r$. Combining this with the recursion relation (1) we obtain

$$\Pi_{k+1}(r)=2(-1)^k\left(\sum_{m=0}^{k}[\ln(1-r)]^m/m!-(1-r)\right) \qquad (3)$$

for $k\geqslant 0$, which implies that $\langle n_k\rangle$ is infinite for all k. Also note that $\lim_{k\to\infty}\Pi_k(r)=\delta(r-1)$. Finally, it is trivial to show that for the Eden model $\Pi_k(r)=1$ and $\langle n_k\rangle=2$ for $k\geqslant 1$.

Next we show that the mean run length converges for a broad class of perturbations to the IP rule. Assume that the choice function satisfies

$$p(1,0)=\varepsilon>0 \qquad (4)$$

i.e. no matter how lopsided the values on the two sides are, there is some non-zero probability of choosing the larger value. Monotonicity then implies that $p(r_1,r_2)>\varepsilon$ for all r_1, r_2. Since $p(r_1,r_2)+p(r_2,r_1)=1$, this also implies $p(r_1,r_2)\leqslant 1-\varepsilon$ so $\Phi(r)\leqslant 1-\varepsilon$. Equation (1) then shows that $\Pi_k(r)\leqslant\varepsilon^{-1}$ and hence $\langle n_\infty\rangle\leqslant\varepsilon^{-2}<\infty$, provided $\Pi_\infty(r)\equiv\lim_{k\to\infty}\Pi_k(r)$ exists. A theorem for Markov chains (e.g. see Feller 1966) guarantees the existence of Π_∞ for choice functions p satisfying (4) which are continuously differentiable. For example, these conditions are satisfied by the choice function used by Martin *et al*, namely $p(r,r')=[1+\exp(k(r-r'))]^{-1}$.

In two dimensions (2D) a simple measure of the radius of a cluster can be obtained by averaging the distance from the centre to the perimeter of the cluster over all directions. This average distance exhibits fractal scaling in IP but scales trivially in the Eden model. A simple one-dimensional (1D) analogue of this radius is $L(t)$, the distance reached at time t on the positive axis. In 1D $\langle L(t)\rangle=t/2$ by symmetry in IP and the Eden model, so the fractal dimension is trivial in both models. However, we shall show that the variance $\sigma^2(t)\equiv\langle L^2(t)\rangle-\langle L(t)\rangle^2$ scales differently in IP and the Eden models. In particular, as $t\to\infty$, $\sigma(t)\sim t$ in IP and $\sigma(t)\sim t^{1/2}$ in the Eden model. To study the crossover between these two regimes, we consider a particular class of choice functions $p(r',r)=\alpha\Theta(r'-r)+(1-\alpha)\Theta(r-r')$ which interpolates continuously between the IP ($\alpha=0$) and Eden ($\alpha=\frac{1}{2}$) limits. We call this the α model. By explicitly evaluating $\sigma(t)$, we show that for any $\alpha>0$ the deviation $\sigma(t)$ crosses over to the Eden scaling behaviour as $t\to\infty$.

Both Π_∞ and $\langle n_\infty\rangle$ can be easily computed for the α model. One obtains

$$\Pi_\infty(r)=\frac{1-2\alpha}{[(2\alpha-1)r+1-\alpha]\ln(\alpha^{-1}-1)}$$

and

$$\langle n_\infty \rangle = \frac{1-2\alpha}{\alpha(1-\alpha)\ln(\alpha^{-1}-1)}.$$

As expected this reduces to $\langle n_\infty \rangle = 2$ in the Eden limit ($\alpha = \frac{1}{2}$), and when $\alpha \to 0$ (IP), $\langle n_\infty \rangle$ diverges like $|\alpha \ln \alpha|^{-1}$. Note the presence of the logarithmic term.

We now proceed to evaluate the deviation $\sigma^2(t)$. The computation begins by constructing a recursion relation for the quantity $g(l, t; r)$, defined as the probability that at time t, the site l with random number r is rejected. This recursion is solved using generating functions and then the moments of the resulting functions are related to the moments of $L(t)$, in particular to $\sigma^2(t)$.

Let

$$P(L, t; r) = f(L+1, t; r) + g(t-L+1, t; r).$$

Then $P(L, t) \equiv \int_0^1 \mathrm{d}r P(L, t; r)$ is the probability that at time t the furthest point reached on the positive x axis is $x = L$, and

$$\sigma^2(t) = \sum_{L=0}^{\infty} P(L, t)L^2 - \left(\sum_{L=0}^{\infty} P(L, t)L \right)^2.$$

To construct a recursion formula for $g(l, t; r)$, note that if site l is rejected at time t, then it was a boundary site after step $t-1$. Then either it was a boundary site before step $t-1$ and was rejected then as well, or it bcame a boundary site at time $t-1$ (site $l-1$ was chosen) so site $-t+l-1$ was rejected. These two possibilities combine to give

$$\begin{aligned} g(l, t; r) &= \int_0^1 \mathrm{d}r' p(r', r)[g(l, t-1; r) + g(-t+l-1, t-1; r')] \\ &= \Phi(r)g(l, t-1; r) + \int_0^1 \mathrm{d}r' p(r', r) g(t-l+1, t-1; r') \end{aligned} \tag{5}$$

where we have used the fact that $g(l, t; r) = g(-l, t; r)$. We define the generating function

$$P(X, t; r) = \sum_{L \geq 0} X^L [g(L+1, t; r) + g(t-L+1, t; r)].$$

Since $g(l, t; r) = 0$ for $l > t+1$,

$$P(X, t; r) = \sum_{L \geq 0} (X^L + X^{t-L}) g(L+1, t; r).$$

We also define

$$Q(X, t; r) = \sum_{L \geq 0} (X^L - X^{t-L}) g(L+1, t; r).$$

The initial conditions are $P(X, 0; r) = X$ and $Q(X, 0; r) = 0$. Using the recursion (5) one obtains

$$\begin{aligned} P(X, t; r) &= \tfrac{1}{2}\Phi(r)[(1+X)P(X, t-1; r) + (1-X)Q(X, t-1; r)] \\ &\quad + \tfrac{1}{2} \int_0^1 p(r', r)[(1+X)P(X, t-1; r') - (1-X)Q(X, t-1; r')] \end{aligned} \tag{6}$$

and

$$Q(X,t;r)=\tfrac{1}{2}\Phi(r)[(1-X)P(X,t-1;r)+(1+X)Q(X,t-1;r)]$$
$$+\tfrac{1}{2}\int_0^1 p(r',r)[(1-X)P(X,t-1;r')-(1+X)Q(X,t-1;r')]. \tag{7}$$

Note that the moments of L are obtained from the function $P(X,t)$ by taking derivatives at $X=1$. For example,

$$\langle L(t)\rangle = X(\partial/\partial X)P(X,t)|_{X=1} \tag{8}$$

and

$$\langle L^2(t)\rangle = X(\partial/\partial X)X(\partial/\partial X)P(X,t)|_{X=1}. \tag{9}$$

Defining further generating functions

$$P(X,Y;r)=\sum_{t\geq 0} Y^t P(X,t;r)$$

and

$$Q(X,Y;r)=\sum_{t\geq 0} Y^t Q(X,t;r)$$

one obtains

$$Y^{-1}[P(X,Y;r)-X]=\tfrac{1}{2}\Phi(r)[(1+X)P(X,Y;r)+(1-X)Q(X,Y;r)]$$
$$+\tfrac{1}{2}\int_0^1 \mathrm{d}r'p(r',r)[(1+X)P(X,Y;r')-(1-X)Q(X,Y;r')] \tag{10}$$

and

$$Y^{-1}Q(X,Y;r)=\tfrac{1}{2}\Phi(r)[(1-X)P(X,Y;r)+(1+X)Q(X,Y;r)]$$
$$+\tfrac{1}{2}\int_0^1 \mathrm{d}r'p(r',r)[(1-X)P(X,Y;r')-(1+X)Q(X,Y;r')]. \tag{11}$$

Subtracting $1-X$ times (11) from $1+X$ times (10) gives Q in terms of P:

$$(1-X)Q(X,Y;r)=(1+X)[P(X,Y;r)-X]-2\Phi(r)XYP(X,Y;r)$$
$$-2XY\int_0^1 \mathrm{d}r'p(r',r)P(X,Y;r'). \tag{12}$$

Differentiating (10) with respect to r gives

$$\frac{1}{Y}\frac{\partial P}{\partial r}=\frac{\Phi}{2}\left((1+X)\frac{\partial P}{\partial r}+(1-X)\frac{\partial Q}{\partial r}\right)+(1-2\alpha)(1+X)P. \tag{13}$$

We now differentiate (12) with respect to r and use (13) to eliminate $\partial Q/\partial r$. The resulting first-order differential equation for P has the solution

$$P(X,Y;r)=\frac{C(X,Y)}{(1-Y\Phi(r))(1-XY\Phi(r))}. \tag{14}$$

The constant of integration $C(X, Y)$ may be evaluated by substitution of (14) into (12) and then into (10). We obtain

$$C(X, Y)=X\left[1+\frac{\alpha(1-\alpha)XY}{(2\alpha-1)(1-X)}\ln\left(\frac{(1-\alpha Y)[1-(1-\alpha)XY]}{(1-\alpha XY)[1-(1-\alpha)Y]}\right)\right]^{-1}. \tag{15}$$

The relation

$$\sum_{t\geq 0} Y^t\langle L^2(t)\rangle=\int_0^1 \mathrm{d}rX\frac{\partial}{\partial X}X\frac{\partial}{\partial X}P(X, Y; r)\bigg|_{X=1}$$

may now be inverted to obtain $\langle L^2(t)\rangle$. Finally, we have the desired result

$$\sigma^2(t)=\left(\frac{1}{4}+\frac{(2\alpha-1)^2}{6\alpha(1-\alpha)}\right)t$$
$$+\frac{2\alpha-1}{6}\left(\frac{\alpha^{t+1}}{(1-\alpha)^2}-\frac{(1-\alpha)^{t+1}}{\alpha^2}-\frac{(2\alpha-1)(1-\alpha+\alpha^2)}{\alpha^2(1-\alpha)^2}\right).$$

For the IP limit ($\alpha=0$), one obtains $\sigma^2(t)=t(t+1)/12$ so that the fluctuations are typically comparable to the size of the system. This result can also be obtained by showing explicitly that $P(L, t)=(t+1)^{-1}$ for IP. All the models with $0<\alpha\leq\frac{1}{2}$, on the other hand, have the limiting behaviour $\sigma^2(t)\sim t$ for large t. For the case of the Eden model ($\alpha=\frac{1}{2}$), this is simply explained: the growth process is analogous to flipping a coin and counting the number of heads. This Bernoulli process has

$$P(L, t)=2^{-t}\binom{t}{L}$$

and therefore $\sigma^2(t)=t/4$.

The crossover between the two regimes can be examined using the scaling variable $z\equiv\alpha(1-\alpha)t$ and considering the large t behaviour of the scaled function $\sigma^2(t)/t^2$ at fixed z. One then obtains the scaling function

$$F(z)=\lim_{\substack{t\to\infty\\ \alpha(1-\alpha)t\ \text{fixed}}}\frac{\sigma^2(t)}{t^2}=\frac{\mathrm{e}^{-z}-1+z}{6z^2}.$$

The function $F(z)$ is finite and non-zero at the origin (the IP limit) but eventually falls to zero like $1/6z$ (Eden behaviour). The crossover length is given by $\xi=[\alpha(1-\alpha)]^{-1}$. Also note that as $\alpha\to 0$ the asymptotic run length diverges like $\langle n_\infty\rangle\sim\xi/\ln\xi$.

Monte Carlo work (Martin *et al* 1984) and phenomenological arguments (Lenormand 1985) have suggested that IP scaling behaviour is disrupted by a wide variety of perturbations. The analytical work presented here bears this out. We have shown that the divergence of the mean run length present in IP is suppressed by a broad class of perturbations to the IP growth rule. We also studied a particular family of models which interpolates between IP ($\alpha=0$) and the Eden model ($\alpha=\frac{1}{2}$). We showed that for any $\alpha>0$, the anomalous scaling behaviour of the variance in IP ($\sigma(t)\sim t$) is replaced by the behaviour found in the Eden model ($\sigma(t)\sim t^{1/2}$) for $t\gg[\alpha(1-\alpha)]^{-1}$.

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